

Exact solutions of Einstein equations

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Abstract

We use a metric of the type Friedmann-Robertson-Walker to obtain new exact solutions of Einstein equations for a scalar and massive field. The solutions have a permanent or transitory inflationary behavior.

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I. INTRODUCTION

There is a great interest in the solutions of Einstein equations due to their success in many of the areas of the physics. Exact solutions make possible a better knowledge of the many problems in several fields, like, for example, Kerr's solution in Astrophysics or Robertson-Walker in Cosmology [1]. The new solutions presented in this paper have an initial or permanent behavior of inflationary nature. Inflation is a very necessary scenario for a primordial universe [2]. This paper is organized as follows. In section II we introduce the equations of Einstein-Klein-Gordon and present their solutions. In section III we present our final considerations.

II. THE EINSTEIN'S FIELD EQUATIONS AND THEIR SOLUTIONS

We use a local basis, with the components of Riemann tensor given by,

$$R^\alpha{}_{\mu\sigma\nu} = \partial_\nu \Gamma^\alpha_{\mu\sigma} - \partial_\sigma \Gamma^\alpha_{\mu\nu} + \Gamma^\eta_{\mu\sigma} \Gamma^\alpha_{\nu\eta} - \Gamma^\eta_{\mu\nu} \Gamma^\alpha_{\sigma\eta}, \quad (1)$$

and components of Ricci tensor

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}. \quad (2)$$

The Einstein's equations, with cosmological constant Λ , are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^2} T_{\mu\nu}, \quad (3)$$

where $T_{\mu\nu}$ is the tensor of momentum-energy for a massive and scalar field [3],

$$T_{\mu\nu} = 2\nabla_\mu \phi \nabla_\nu \phi - g_{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi + m^2 g_{\mu\nu} \phi^2. \quad (4)$$

The massive scalar field obeys to the following motion equation

$$\partial_\mu \{ \sqrt{-g} g^{\mu\nu} \partial_\nu \phi \} + m^2 \sqrt{-g} \phi = 0. \quad (5)$$

We use (+, -, -, -) signature convention and one line element of the type Friedmann-Robertson-Walker, given by

$$ds^2 = dt^2 - \frac{d\sigma^2 e^g}{(1 + Br^2)^2} \quad (6)$$

where $d\sigma^2$ is the three-dimensional Euclidian line element. In this paper we use $A = 8\pi G/c^2$, $c = 1$, $B = k/4a^2$ and with $k = 0$, $k = 1$ and $k = -1$. We also have a^2 a constant.

Due to isotropy and homogeneity of (6), the equations (3) and (5) assume the following forms

$$\left(\frac{d\phi}{dt}\right)^2 + m^2\phi^2 + \frac{\Lambda}{A} - \frac{3}{4A}\left(\frac{dg}{dt}\right)^2 - \frac{12B}{A}e^{-g} = 0, \quad (7)$$

$$\left(\frac{d\phi}{dt}\right)^2 - m^2\phi^2 - \frac{\Lambda}{A} + \frac{3}{4A}\left(\frac{dg}{dt}\right)^2 + \frac{1}{A}\frac{d^2g}{dt^2} + \frac{4B}{A}e^{-g} = 0 \quad (8)$$

and

$$\frac{d^2\phi}{dt^2} + \frac{3}{2}\frac{dg}{dt}\frac{d\phi}{dt} + m^2\phi = 0 \quad (9)$$

where $\phi = \phi(t)$ and (9) is the motion equation of the field. We will present now the three exact solutions of the system formed by the equations (7), (8) and (9). We consider, initially, the case $B = 0$, followed by $B > 0$ and later by $B < 0$.

For $B = 0$ the field is given by

$$\phi = \frac{\in mt}{\sqrt{3A}} + b \quad (10)$$

with $\in = \pm 1$ and b an arbitrary constant.

In this case the cosmological constant obeys the condition

$$\Lambda = -\frac{m^2}{3}, \quad (11)$$

a negative value, associated with the mass of the scalar field. The corresponding line element will be

$$ds^2 = dt^2 - d\sigma^2 e^{[-2\in mb(\sqrt{\frac{A}{3}}t - \frac{m^2}{3}t^2)],} \quad (12)$$

that, for special conditions has a transitory inflationary nature followed by a contraction as we will see in section III.

Let us consider the condition $B > 0$. In this case the cosmological constant Λ is proportional to the square of the mass, but is a positive constant, given by,

$$\Lambda = \frac{3m^2}{2}. \quad (13)$$

For the field, we obtain the following result,

$$\phi = \frac{\epsilon}{m} \sqrt{\frac{8B}{A}} e^{(-\frac{\epsilon mt}{\sqrt{2}})} \quad (14)$$

The line element of the metric will be,

$$ds^2 = dt^2 - d\sigma^2 \frac{e^{(\epsilon\sqrt{2}mt)}}{(1 + Br^2)^2}. \quad (15)$$

Let us see now $B < 0$.

In this case we have to be more careful in the analyse of the results having it in mind, we obtain, for the the field, the expression

$$\phi = \epsilon \sqrt{\frac{12B}{\Lambda A}} e^{(-\frac{\epsilon mt}{\sqrt{2}})} \quad (16)$$

The cosmological constant Λ , for the above conditions, will also be given by (13). Substituting (13) in (16) the field assume the following form,

$$\phi = \epsilon \sqrt{\frac{8B}{Am^2}} e^{(-\frac{\epsilon mt}{\sqrt{2}})}, \quad (17)$$

and the correspondent line element will be,

$$ds^2 = dt^2 - d\sigma^2 \frac{e^{(\epsilon\sqrt{2}mt)}}{(1 + Br^2)^2}. \quad (18)$$

III. CONCLUSIONS

We will now make some considerations about the results of the previous section. In the case $B = 0$ will specialize the constants ϵ and b so that we have an initial inflationary phase. Let us take $\epsilon = -1$ and $b > 0$. The line element (12) will now have a dominant inflationary phase to $t < (6b/m) \sqrt{A/3}$, followed by a contraction for $t > (6b/m) \sqrt{A/3}$. The field, given

by (10), will be positive for $t < (b/m) \sqrt{3A}$ and negative for $t > (b/m) \sqrt{3A}$. If we assume that $\epsilon = +1$ and $b < 0$, the line element will not change and will have an initial dominant inflationary phase to $t < -(6b/m) \sqrt{A/3}$ and one contracted phase to $t > -(6b/m) \sqrt{A/3}$. In this two situations, as it can be seen by motion equation (9), the field will not have a dynamic character, but associated with the cosmological constant Λ , both will have a decisive dynamic character and they suggest that their interactions with the other fields of the nature can be thought as fields in a background field. As a consequence, the space-time of the metric (12) can be thought as a reservoir of energy, and as an infinite arena where the other fields of nature interact [4]. For $B > 0$, we will have a finite space-time, with an inflationary expansion that could be weakened by a processes of mass generation of the gauge fields, and greatly slowed down by the gravitational interaction with the cosmic fluid. It is interesting to observe that in the present days the value of Λ is estimated in 10^{-54} cm^{-2} . This means that the mass of the field is very small [5]. The expansion would be very slow and we could think about particles confined in a background field with topology $\mathbb{R} \times \mathbb{S}^3$. Let us now consider the case $B < 0$. With this condition we will have an imaginary field, given by (17) in an infinite universe (18), since the mass of the field was real and positive. If the mass is considered as an imaginary constant the field and the metric (18) will be complex. In the two situations we will have problems with the momentum-energy tensor. We conclude this paper admitting that part of this section is speculative because we have not made applications of our results, but in some cases, it will be possible to accomplish estimates and confront them with the observational data, and conclude if the new solutions are or not physically useful.

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